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MULTILINEAR COMMUTATORS FOR FRACTIONAL INTEGRALS IN NON-HOMOGENEOUS SPACES

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Abstract

Under the assumption that μ is a non-doubling measure on $\mathbb{R}^d$, the authors obtain the (L^p, L^q)-boundedness and the weak type endpoint estimate for the multilinear commutators generated by fractional integrals with $\mathrm{RBMO}(\mu)$ functions of Tolsa or with $\mathrm{Osc}_{\exp L^r}(\mu)$ functions for $r \geq 1$, where $\mathrm{Osc}_{\exp L^r}(\mu)$ is a space of Orlicz type satisfying that $\mathrm{Osc}_{\exp L^r}(\mu) = \mathrm{RBMO}(\mu)$ if $r = 1$ and $\mathrm{Osc}_{\exp L^r}(\mu) \subset \mathrm{RBMO}(\mu)$ if $r > 1$.

1. Introduction

Let μ be a positive Radon measure on $\mathbb{R}^d$ which only satisfies the following growth condition

$$\mu(B(x,r)) \leq Cr^n \tag{1.1}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, where n is a fixed number and $0 < n \leq d$. The doubling condition on μ, namely, there exists some positive constant C such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r))$$

for all $x \in \operatorname{supp}\mu$ and $r > 0$, is an essential assumption in the classical theory of harmonic analysis. But recently, many classical results have been proved still valid if the underlying measure μ is substituted by a non-doubling Radon measure as in (1.1); see [**10**], [**17**], [**18**], [**11**], [**19**], [**20**], [**21**], [**22**], [**23**], [**12**], [**13**], [**8**], [**6**], [**14**], [**2**], [**4**], [**5**], [**7**] and their references.

The purpose of this paper is to prove the (L^p, L^q)-boundedness and the weak type estimate for any multilinear commutator generated by the fractional integral I_α related to a measure μ as in (1.1) for $0 < \alpha < n$

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with any RBMO(μ) function of Tolsa in [**20**] or with any $\mathrm{Osc}_{\exp L^r}(\mu)$ function for $r \geq 1$, motivated by [**16**], [**3**] for commutators on $\mathbb{R}^d$ in the case that μ is the d-dimensional Lebesgue measure there. Here, for $0 < \alpha < n$ and all $x \in \mathrm{supp}\,(\mu)$,

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{n-\alpha}} f(y)\, d\mu(y).$$

The behavior of such fractional integrals on a metric space was recently studied by García-Cuerva and Gatto in [**4**]. Chen and Sawyer [**2**] showed that the first order commutator generated by I_α and RBMO(μ) function enjoys the same (L^p, L^q) mapping properties as in the case that μ is the Lebesgue measure. However, it seems that the argument used in [**2**] does not apply to the multilinear commutator here, we will employ some ideas used in [**2**] and some new ideas different from ones used in [**2**].

Before stating our results, let us introduce some notation and recall some definitions. Throughout this paper, we only consider closed cubes with sides parallel to coordinate axes. Let $\alpha > 1$ and $\beta > \alpha^n$. We say that a cube Q is a (α, β)-doubling cube if $\mu(\alpha Q) \leq \beta \mu(Q)$, where αQ denotes the cube with the same center as Q and $l(\alpha Q) = \alpha l(Q)$. In what follows, for definiteness, if α and β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube. Especially, for any given cube Q, we denote by $\widetilde{Q}$ the smallest doubling cube which contains Q and has the same center as Q.

Let $0 \leq \gamma < n$. Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$K^{(\gamma)}_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \left[\frac{\mu(2^k Q)}{l(2^k Q)^n}\right]^{1-\gamma/n},$$

where $N_{Q,R}$ is the smallest positive integer k such that $l(2^k Q) \geq l(R)$. If $\gamma = 0$, then we denote $K^{(0)}_{Q,R}$ by $K_{Q,R}$. Tolsa in [**20**] first introduced the concept of $K_{Q,R}$ and gave its several useful properties. Chen and Sawyer in [**2**] introduced $K^{(\gamma)}_{Q,R}$ and established some properties on $K^{(\gamma)}_{Q,R}$ similar to those on $K_{Q,R}$.

Using $K_{Q,R}$, Tolsa in [**20**] introduced the space RBMO(μ) with the non-doubling measure μ, which is proved to be a good substitute of the classical space BMO in this case.

Definition 1.1. Let $\rho > 1$ be some fixed constant. We say that a function $f \in L^1_{\mathrm{loc}}(\mu)$ is in RBMO(μ) if there exists some constant $C > 0$ such that for any cube Q centered at some point of $\mathrm{supp}\,(\mu)$,

$$\frac{1}{\mu(\rho Q)} \int_Q \left| f(y) - m_{\widetilde{Q}}(f) \right| d\mu(y) \leq C$$

and

$$|m_Q(f) - m_R(f)| \leq CK_{Q,R}$$

for any two doubling cube $Q \subset R$, where $m_Q(f)$ denotes the mean of f over cube Q. The minimal constant C as above is the RBMO(μ) norm of f and is denoted by $\|f\|_*$.

It has been shown in [**20**] that the definition of RBMO(μ) is independent of chosen constant ρ. In this paper, we need to choose different ρ in the proof of Theorem 1.1 and Theorem 1.2 below, respectively.

For any $m \in \mathbb{N}$, $0 < \alpha < n$ and $b_i \in$ RBMO(μ), $i = 1, 2, \ldots, m$, the multilinear commutator, $I_{\alpha;\, b_1,\ldots,b_m}$, is defined by $[b_m, \ldots, [b_2, [b_1, I_\alpha]] \ldots]$, that is,

$$I_{\alpha;\, b_1,b_2,\ldots,b_m} f(x) = \int_{\mathbb{R}^d} \prod_{j=1}^{m} [b_j(x) - b_j(y)] \frac{f(y)}{|x-y|^{n-\alpha}}\, d\mu(y). \tag{1.2}$$

When $m = 1$, the operator $I_{\alpha;\, b_1,\ldots,b_m}$ is just the commutator $[b_1, I_\alpha]$, which is a variant on the non-doubling measure of the classical commutator studied by Chanillo in [**1**]. In [**16**], Pérez and Trujillo-González studied the boundedness of multilinear operators of this type generated by Calderón-Zygmund operators with BMO$(\mathbb{R}^d)$ functions or with $\mathrm{Osc}_{\exp L^r}$ functions for $r \geq 1$ in the case that μ is the d-dimensional Lebesgue measure. An extensive study of multilinear operators of this type can also be founded in [**9**]. In [**7**], the authors obtained the corresponding results of multilinear operators generated by Calderón-Zygmund operators with RBMO(μ) functions or with $\mathrm{Osc}_{\exp L^r}(\mu)$ functions for $r \geq 1$ in the case that μ is a non doubling measure. The multilinear commutator $I_{\alpha;\, b_1,\ldots,b_m}$ can be regarded as a natural variant of these multilinear operators in [**9**], [**16**], [**7**].

Chen and Sawyer in [**2**] proved that $I_{\alpha;\, b_1}$ as in (1.2) is bounded from $L^p(\mu)$ to $L^q(\mu)$ provided that $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. For $m \geq 2$, a conclusion similar to that for the case $m = 1$ in [**2**] can be obtained as follows.

Theorem 1.1. *Let $m \in \mathbb{N}$ and $b_j \in$ RBMO(μ), $j = 1, 2, \ldots, m$. For $\alpha \in (0, n)$, let $I_{\alpha;\, b_1,b_2,\ldots,b_m}$ be as in* (1.2). *Then there exists a constant $C > 0$ such that for all $f \in L^p(\mu)$,*

$$\|I_{\alpha;\, b_1,b_2,\ldots,b_m} f\|_{L^q(\mu)} \leq C \prod_{j=1}^{m} \|b_j\|_* \|f\|_{L^p(\mu)},$$

where $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$.

Remark 1.1. It is well-known that the commutator $[b, I_\alpha]$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ provided $1 < p < d/\alpha$ and $1/q = 1/p - \alpha/d$, if and only if $b \in \mathrm{BMO}(\mathbb{R}^d)$ if the measure μ is the d-dimensional Lebesgue measure; see [**1**]. However, it is still open if $b \in \mathrm{RBMO}(\mu)$ is a necessary condition for the $(L^p(\mu), L^q(\mu))$-boundedness of the multilinear commutators $I_{\alpha;\, b_1,\ldots,b_m}$ on non doubling measures.

To consider the endpoint case of Theorem 1.1, we introduce the following function space, which is a variant with a non-doubling measure of the space $\mathrm{Osc}_{\exp L^r}$ in [**16**].

Definition 1.2. For $r \geq 1$, a locally integrable function f is said to belong to the space $\mathrm{Osc}_{\exp L^r}(\mu)$ if there is a constant $C_1 > 0$ such that

(i) for any Q,

$$\|f - m_{\widetilde{Q}}(f)\|_{\exp L^r, Q, \mu/\mu(2Q)}$$
$$= \inf \left\{ \lambda > 0 : \frac{1}{\mu(2Q)} \int_Q \exp\left(\frac{|f(x) - m_{\widetilde{Q}}(f)|}{\lambda} \right)^r d\mu(x) \leq 2 \right\} \leq C_1,$$

(ii) for any doubling cubes $Q_1 \subset Q_2$,

$$|m_{Q_1}(f) - m_{Q_2}(f)| \leq C_1 K_{Q_1,Q_2}.$$

The minimal constant C_1 satisfying (i) and (ii) is the $\mathrm{Osc}_{\exp L^r}(\mu)$ norm of f and is denoted by $\|f\|_{\mathrm{Osc}_{\exp L^r}(\mu)}$.

Obviously, for any $r \geq 1$, $\mathrm{Osc}_{\exp L^r(\mu)} \subset \mathrm{RBMO}(\mu)$. Moreover, from John-Nirenberg's inequality in [**20**] (see also Lemma 3.1 below), it follows that $\mathrm{Osc}_{\exp L(\mu)} = \mathrm{RBMO}(\mu)$. We remark that it was pointed by Pérez and Trujillo-González in [**16**] that if μ is the d-dimensional Lebesgue measure in $\mathbb{R}^d$, the counterpart in [**16**] of the space $\mathrm{Osc}_{\exp L^r}(\mu)$ when $r > 1$ is a proper subspace of the classical space $\mathrm{BMO}(\mathbb{R}^d)$. However, it is still unknown if the space $\mathrm{Osc}_{\exp L^r}(\mu)$ is a proper subspace of the space $\mathrm{RBMO}(\mu)$ when μ is a non-doubling measure?

To state the weak type estimate for the multilinear commutator $I_{\alpha;\, b_1,\ldots,b_m}$, we still need to introduce the following notation. For $1 \leq j \leq m$, we denote by C_j^m the family of all finite subset $\sigma = \{\sigma(1), \ldots, \sigma(j)\}$ of $\{1, 2, \ldots, m\}$ with j different elements. For any $\sigma \in C_j^m$, the complementary sequence σ' is given by $\sigma' = \{1, 2, \ldots, m\} \backslash \sigma$. For any $\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(j)\} \in C_j^m$, we write for any m-tuple $r = (r_1, r_2, \ldots, r_m)$, $1/r_\sigma = 1/r_{\sigma(1)} + \cdots + 1/r_{\sigma(j)}$ and $1/r_{\sigma'} = 1/r - 1/r_\sigma$, where $1/r = 1/r_1 + \cdots + 1/r_m$.

The following weak type estimate is the main result of this paper, which is new even when $m = 1$, namely, Theorem 1.2 is also new even for the commutator of the first order, $[b_1, I_\alpha]$.

Theorem 1.2. *Let* $0 < \alpha < n$, $q_0 = n/(n-\alpha)$, $m \in \mathbb{N}$, $r_i \ge 1$ *and* $b_i \in \mathrm{Osc}_{\exp L^{r_i}(\mu)}$ *for* $i = 1, 2, \ldots, m$. *Let* $I_{\alpha;\, b_1,\ldots,b_m}$ *be as in* (1.2). *Then there exists a constant* $C > 0$ *such that for all bounded functions* f *with compact support and all* $\lambda > 0$,

$$\mu\left(\left\{x \in \mathbb{R}^d : |I_{\alpha;\, b_1,\ldots,b_m} f(x)| > \lambda\right\}\right) \le C\left[\Phi_{1/r}\left(\prod_{j=1}^{m} \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}(\mu)}\right)\right]^{q_0} \times \left\{\sum_{j=0}^{m} \sum_{\sigma \in C_j^m} \Phi_{1/r_\sigma}\left[\|\Phi_{1/r_{\sigma'}}(\lambda^{-1}|f|)\|_{L^1(\mu)}\right]\right\}^{q_0},$$

where $\Phi_s(t) = t \log^s(2+t)$ *for all* $t > 0$ *and* $s > 0$.

In what follows, C denotes a constant that is independent of main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$. For $A \sim B$, we mean that there is a constant $C > 0$ such that $C^{-1}B \le A \le CB$.

Let $1 \le s < \infty$. For a μ-locally integrable function f and a cube Q, we define

$$\|f\|_{L\log^s L, Q, \mu/\mu(2Q)} = \inf\left\{\lambda > 0 : \frac{1}{\mu(2Q)} \int_Q \frac{|f(x)|}{\lambda} \log^s\left(2 + \frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\}$$

and

$$\|f\|_{\exp L^s, Q, \mu/\mu(2Q)} = \inf\left\{\lambda > 0 : \frac{1}{\mu(2Q)} \int_Q \exp\left(\frac{|f(x)|}{\lambda}\right)^s d\mu(x) \le 2\right\}.$$

Then the generalized Hölder inequality

$$\text{(1.3)} \quad \frac{1}{\mu(2Q)} \int_Q |f(x) b_1(x) \cdots b_m(x)|\, d\mu(x) \le C\|f\|_{L\log^{1/r} L, Q, \mu/\mu(2Q)} \|b_1\|_{\exp L^{r_1}, Q, \mu/\mu(2Q)} \cdots \|b_m\|_{\exp L^{r_m}, Q, \mu/\mu(2Q)}$$

holds for μ-locally integrable functions f and b_i, $i = 1, 2, \dots, m$, and any cube Q, provided that $r_1, r_2, \dots, r_m \geq 1$, $1/r = 1/r_1 + \cdots + 1/r_m$. If μ is the d-dimensional Lebesgue measure, (1.3) was proved in [**16**]. It is easy to see that the proof in [**16**] still works in non-homogeneous spaces.

2. Proof of Theorem 1.1

Before we begin to prove Theorem 1.1, we state one equivalent norm for the space RBMO(μ) and some lemmas which play important roles in the proof.

Let $\rho > 1$. For any given function $b \in \mathrm{RBMO}(\mu)$, there exists some constant $C_2 > 0$ and a collection of numbers $\{b_Q\}_Q$, namely, for each cube Q, there exists a number $b_Q \in \mathbb{R}$, such that

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |b(y) - b_Q| \, d\mu(y) \leq C_2$$

and

$$|b_Q - b_R| \leq C_2 K_{Q,R}$$

for any two cubes $Q \subset R$. Let $\|b\|_{**} = \inf\{C_2\}$, where the infimum is taken over all $C_2 > 0$ as above. Then there is a constant $C > 0$ such that for all $b \in \mathrm{RBMO}(\mu)$,

$$C^{-1}\|b\|_* \leq \|b\|_{**} \leq C\|b\|_*; \tag{2.1}$$

see [**20**].

Lemma 2.1 ([**6**], [**4**]). *Let $1 < p < \infty$, $1/q = 1/p - \alpha/n$ and $0 < \alpha < n$. Then*

$$\|I_\alpha f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mu)}.$$

Lemma 2.2 ([**2**]). *For $\eta > 1$ and $0 \leq \beta < n/p$, we define the non centered maximal operator*

$$M^{(\beta)}_{p,(\eta)} f(x) = \sup_{x \in Q} \left[\frac{1}{\mu(\eta Q)^{1-\beta p/n}} \int_Q |f(y)|^p \, d\mu(y) \right]^{1/p}.$$

When $\beta = 0$, we denote $M^{(\beta)}_{p,(\eta)}$ simply by $M_{p,(\eta)}$. If $p < r < n/\beta$ and $1/q = 1/r - \beta/n$, then

$$\|M^{(\beta)}_{p,(\eta)} f\|_{L^q(\mu)} \leq C\|f\|_{L^r(\mu)},$$

and if $q > p$, then

$$\|M_{p,(\eta)} f\|_{L^q(\mu)} \leq C\|f\|_{L^q(\mu)}.$$

Moreover, we also need to introduce another variant sharp maximal operator $M^{\sharp,(\beta)}f$ defined by

$$M^{\sharp,(\beta)}f(x) = \sup_{x\in Q} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{\widetilde{Q}}(f)|\, d\mu(y) + \sup_{\substack{x\in Q\subset R\\ Q,R \text{ doubling}}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}^{(\beta)}},$$

and non centered doubling maximal operator N:

$$Nf(x) = \sup_{\substack{x\in Q\\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y).$$

By the Lebesgue differentiation theorem, it is easy to see that for any $f \in L^1_{\mathrm{loc}}(\mu)$,

$$|f(x)| \le Nf(x) \tag{2.2}$$

for μ-a.e. $x \in \mathbb{R}^d$; see [**20**] and [**2**].

Lemma 2.3 ([**2**]). *Let $f \in L^1_{loc}(\mu)$ with*

$$\int_{\mathbb{R}^d} f(y)\, d\mu(y) = 0$$

if $\|\mu\| < \infty$. For $1 < p < \infty$, if $\inf(1, Nf) \in L^p(\mu)$, then for $0 \le \beta < n$, we have

$$\|Nf\|_{L^p(\mu)} \le C\|M^{\sharp,(\beta)}f\|_{L^p(\mu)}.$$

Proof of Theorem 1.1: For simplicity, we only consider the case of $m=2$. If $m \ge 2$, we can deduce the conclusion of the theorem by induction on m. We leave the details to the reader; see also the proof of Theorem 1.2 below.

For all $r \in (1, n/\alpha)$, we will prove the following sharp maximal function estimate

$$\begin{aligned} M^{\sharp,(\alpha)}(I_{\alpha;\, b_1,b_2}f)(x) \le C\|b_1\|_*\|b_2\|_* \Big\{ & M^{(\alpha)}_{r,(9/8)}f(x) \\ & + M_{r,(3/2)}\left[I_\alpha\left(|f|\right)\right](x) \Big\} \\ & + C\|b_1\|_* M_{r,(3/2)}(I_{\alpha;\, b_2}f)(x) \\ & + C\|b_2\|_* M_{r,(3/2)}(I_{\alpha;\, b_1}f)(x). \end{aligned} \tag{2.3}$$

Then choose r such that $1 < r < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. By (2.3), (2.2), Lemma 2.1, Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}
\|I_{\alpha;\,b_1,b_2} f\|_{L^q(\mu)} &\le C\|N(I_{\alpha;\,b_1,b_2} f)\|_{L^q(\mu)} \\
&\le C\|M^{\sharp,(\alpha)}(I_{\alpha;\,b_1,b_2} f)\|_{L^q(\mu)} \\
&\le C\|b_1\|_*\|b_2\|_* \left\{ \left\| M^{(\alpha)}_{r,(9/8)} f \right\|_{L^q(\mu)} \right. \\
&\qquad\qquad \left. + \ \left\| M_{r,(3/2)} \left[I_\alpha \left(|f| \right) \right] \right\|_{L^q(\mu)} \right\} \\
&\quad + C\|b_1\|_* \left\| M_{r,(3/2)}(I_{\alpha;\,b_2} f) \right\|_{L^q(\mu)} \\
&\quad + C\|b_2\|_* \left\| M_{r,(3/2)}(I_{\alpha;\,b_1} f) \right\|_{L^q(\mu)} \\
&\le C\|b_1\|_*\|b_2\|_*\|f\|_{L^p(\mu)},
\end{aligned}$$

which is the desired conclusion.

For $j = 1, 2$, let $\{b^j_Q\}_Q$ be a family of numbers satisfying

$$\int_Q |b_j(y) - b^j_Q|\, d\mu(y) \le 2\mu(2Q)\|b_j\|_{**}$$

for any cube Q, and

$$|b^j_Q - b^j_R| \le 2K_{Q,R}\|b\|_{**}$$

for all cubes $Q \subset R$. For any cube Q, we let

$$h_Q = m_Q \left(I_\alpha \left[(b_1 - b^1_Q)(b_2 - b^2_Q) f \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q} \right] \right).$$

To establish (2.3), it suffices to verify that

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}Q)} \int_Q |I_{\alpha;\,b_1,b_2} f(y) - h_Q|\, d\mu(y) \\
&\quad \le C\|b_1\|_*\|b_2\|_* \left[M^{(\alpha)}_{r,(9/8)} f(x) + M_{r,(3/)}(I_\alpha f)(x) \right] \\
&\qquad + C\|b_1\|_* M_{r,(3/2)}(I_{\alpha;\,b_2} f)(x) + C\|b_2\|_* M_{r,(3/2)}(I_{\alpha;\,b_1} f)(x)
\end{aligned} \tag{2.4}$$

holds for any cube Q and any $x \in Q$, and

$$
\begin{aligned}
|h_Q - h_R| \leq C\|b_1\|_*\|b_2\|_* K_{Q,R}^2 \Big\{ & K_{Q,R}^{(\alpha)} M_{r,(9/8)}^{(\alpha)} f(x) \\
& + M_{r,(3/2)} \left[I_\alpha(|f|) \right](x) \Big\} \\
& + C\|b_1\|_* K_{Q,R} M_{r,(3/2)}(I_{\alpha;\, b_2} f)(x) \\
& + C\|b_2\|_* K_{Q,R} M_{r,(3/2)}(I_{\alpha;\, b_1} f)(x)
\end{aligned}
\tag{2.5}
$$

for any cubes $Q \subset R$ with $x \in Q$, where Q is an arbitrary cube and R is a doubling cube.

By a method similar to that in [**2**], from (2.4) and (2.5), it is then easy to deduce the sharp maximal function estimate (2.3).

To obtain the estimate (2.4) for any fixed Q and $x \in Q$, write

$$
\begin{aligned}
& \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |I_{\alpha;\, b_1, b_2} f(y) - h_Q| \, d\mu(y) \\
& \quad \leq \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q \prod_{j=1,2} \left| b_j(y) - b_Q^j \right| |I_\alpha f(y)| \, d\mu(y) \\
& \qquad + \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q \left| b_1(y) - b_Q^1 \right| \left| I_{\alpha;\, b_2} f(y) \right| \, d\mu(y) \\
& \qquad + \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q \left| b_2(y) - b_Q^2 \right| \left| I_{\alpha;\, b_1} f(y) \right| \, d\mu(y) \\
& \qquad + \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q \left| I_\alpha \left[(b_1 - b_Q^1)(b_1 - b_Q^1) f \right](y) - h_Q \right| \, d\mu(y) \\
& \quad = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.
\end{aligned}
$$

The Hölder inequality and Corollary 3.5 in [**20**] yield that for $1 < r < n/\alpha$,

$$
\begin{aligned}
\mathrm{I}_1 \leq & \left\{ \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q \left| (b_1(y) - b_Q^1)(b_2(y) - b_Q^2) \right|^{r'} \, d\mu(y) \right\}^{1/r'} \\
& \times \left\{ \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |I_\alpha f(y)|^r \, d\mu(y) \right\}^{1/r} \\
\leq & \; C\|b_1\|_*\|b_2\|_* M_{r,(3/2)}(I_\alpha f)(x),
\end{aligned}
\tag{2.6}
$$

and

$$\begin{aligned}\mathrm{I}_2 &\le \left[\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|b_1(y)-b_Q^1\right|^{r'}\,d\mu(y)\right]^{1/r'}\\ &\qquad\times\left[\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q |I_{\alpha;\,b_2}f(y)|^r\,d\mu(y)\right]^{1/r}\\ &\le C\|b_1\|_* M_{r,(3/2)}(I_{\alpha;\,b_2}f)(x).\end{aligned}\tag{2.7}$$

An estimate similar to that for I_2 tells us that

$$\mathrm{I}_3 \le C\|b_2\|_* M_{r,(3/2)}\left(I_{\alpha;\,b_1}f\right)(x).\tag{2.8}$$

To estimate I_4, let $f_1 = f\chi_{\frac{4}{3}Q}$ and $f_2 = f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}$. Decompose I_4 into

$$\begin{aligned}\mathrm{I}_4 &\le \frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|I_\alpha\left[(b_1-b_Q^1)(b_2-b_Q^2)f_1\right](y)\right|\,d\mu(y)\\ &\quad+\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q \left|I_\alpha\left[(b_1-b_Q^1)(b_2-b_Q^2)f_2\right](y)-h_Q\right|\,d\mu(y)\\ &= \mathrm{E}+\mathrm{F}.\end{aligned}$$

Let $s=\sqrt{r}$ and $1/\nu = 1/s-\alpha/n$. From the Hölder inequality, (2.1) and Lemma 2.1, it follows that

$$\begin{aligned}\mathrm{E} &\le C\frac{\mu(Q)^{1-1/\nu}}{\mu\left(\frac{3}{2}Q\right)}\left\|I_\alpha\left[(b_1-b_Q^1)(b_2-b_Q^2)f_1\right]\right\|_{L^\nu(\mu)}\\ &\le C\frac{\mu(Q)^{1-1/\nu}}{\mu\left(\frac{3}{2}Q\right)}\left\|(b_1-b_Q^1)(b_2-b_Q^2)f_1\right\|_{L^s(\mu)}\\ &\le C\|b_1\|_*\|b_2\|_* M^{(\alpha)}_{r,(9/8)}f(x).\end{aligned}\tag{2.9}$$

To estimate F, by the Hölder inequality, we first have that for $y_1, y_2 \in Q$,

(2.10)

$$
\begin{aligned}
&\left|I_\alpha\left[(b_1 - b_Q^1)(b_2 - b_Q^2)f_2\right](y_1) - I_\alpha\left[(b_1 - b_Q^1)(b_2 - b_Q^2)f_2\right](y_2)\right| \\
&\leq C\int_{\mathbb{R}^d\setminus\frac{4}{3}Q} \frac{|y_1 - y_2|}{|z - y_1|^{n+1-\alpha}} \left|(b_1(z) - b_Q^1)(b_2(z) - b_Q^2)\right| |f(z)|\, d\mu(z) \\
&\leq C\sum_{k=1}^{\infty}\int_{2^k\frac{4}{3}Q\setminus 2^{k-1}\frac{4}{3}Q} \frac{l(Q)}{|z - y_1|^{n+1-\alpha}} \\
&\qquad\qquad \times \left|(b_1(z) - b^1_{2^k\frac{4}{3}Q})(b_2(z) - b^2_{2^k\frac{4}{3}Q})\right| |f(z)|\, d\mu(z) \\
&\quad + C\sum_{k=1}^{\infty}\left|b_Q^1 - b^1_{2^k\frac{4}{3}Q}\right| \int_{2^k\frac{4}{3}Q\setminus 2^{k-1}\frac{4}{3}Q} \frac{l(Q)}{|z - y_1|^{n+1-\alpha}} \\
&\qquad\qquad \times \left|b_2(z) - b^2_{2^k\frac{4}{3}Q}\right| |f(z)|\, d\mu(z) \\
&\quad + C\sum_{k=1}^{\infty}\left|b_Q^2 - b^2_{2^k\frac{4}{3}Q}\right| \int_{2^k\frac{4}{3}Q\setminus 2^{k-1}\frac{4}{3}Q} \frac{l(Q)}{|z - y_1|^{n+1-\alpha}} \\
&\qquad\qquad \times \left|b_1(z) - b^1_{2^k\frac{4}{3}Q}\right| |f(z)|\, d\mu(z) \\
&\quad + C\sum_{k=1}^{\infty}\left|(b_Q^1 - b^1_{2^k\frac{4}{3}Q})(b_Q^2 - b^2_{2^k\frac{4}{3}Q})\right| \int_{2^k\frac{4}{3}Q\setminus 2^{k-1}\frac{4}{3}Q} \frac{l(Q)}{|z - y_1|^{n+1-\alpha}} |f(z)|\, d\mu(z) \\
&\leq C\sum_{k=1}^{\infty} k^2 2^{-k}\|b_1\|_*\|b_2\|_* \left[\frac{1}{\mu(2^k\frac{3}{2}Q)^{1-\alpha r/n}}\int_{2^k\frac{4}{3}Q} |f(z)|^r\, d\mu(z)\right]^{1/r} \\
&\leq C\|b_1\|_*\|b_2\|_* M^{(\alpha)}_{r,(9/8)}f(x),
\end{aligned}
$$

where we used the estimate

$$|b_j - b^j_{2^k\frac{4}{3}Q}| \leq Ck\|b_j\|_{**} \leq Ck\|b_j\|_*$$

for $j = 1, 2$. From (2.10) and the choice of h_Q, it follows that

$$\left|I_\alpha\left[(b_1 - b_Q^1)(b_2 - b_Q^2)f_2\right](y_1) - h_Q\right| \leq C\|b_1\|_*\|b_2\|_* M^{(\alpha)}_{r,(9/8)}f(x).$$

Therefore,

$$\mathrm{F} \leq C\|b_1\|_*\|b_2\|_* M^{(\alpha)}_{r,(9/8)} f(x). \tag{2.11}$$

The estimates (2.9) and (2.11) indicate

$$\mathrm{I}_4 \leq C\|b_1\|_*\|b_2\|_* M^{(\alpha)}_{r,(9/8)} f(x). \tag{2.12}$$

Combining (2.6), (2.7), (2.8) and (2.12) yields (2.4).

We now check (2.5) for chosen $\{h_Q\}_Q$ as above. Consider two cubes $Q \subset R$ with $x \in Q$ and a doubling cube R. Denote $N_{Q,R} + 1$ simply by N. Write

$$\begin{aligned}
|h_Q - h_R| &= \Big| m_Q\left(I_\alpha\left[(b_1 - b_Q^1)(b_2 - b_Q^2) f \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}\right]\right) \\
&\qquad\qquad - m_R\left(I_\alpha\left[(b_1 - b_R^1)(b_2 - b_R^2) f \chi_{\mathbb{R}^d \setminus \frac{4}{3}R}\right]\right)\Big| \\
&\leq \big| m_R\left(I_\alpha\left[(b_1 - b_Q^1)(b_2 - b_Q^2) f \chi_{\mathbb{R}^d \setminus 2^N Q}\right]\right) \\
&\qquad\qquad - m_Q\left(I_\alpha\left[(b_1 - b_Q^1)(b_2 - b_Q^2) f \chi_{\mathbb{R}^d \setminus 2^N Q}\right]\right)\big| \\
&\quad + \big| m_R\left(I_\alpha\left[(b_1 - b_R^1)(b_2 - b_R^2) f \chi_{\mathbb{R}^d \setminus 2^N Q}\right]\right) \\
&\qquad\qquad - m_R\left(I_\alpha\left[(b_1 - b_Q^1)(b_2 - b_Q^2) f \chi_{\mathbb{R}^d \setminus 2^N Q}\right]\right)\big| \\
&\quad + \Big| m_Q\left(I_\alpha\left[(b_1 - b_Q^1)(b_2 - b_Q^2) f \chi_{2^N Q \setminus \frac{4}{3}Q}\right]\right)\Big| \\
&\quad + \Big| m_R\left(I_\alpha\left[(b_1 - b_R^1)(b_2 - b_R^2) f \chi_{2^N Q \setminus \frac{4}{3}R}\right]\right)\Big| \\
&= \mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3 + \mathrm{L}_4.
\end{aligned}$$

Similar to the estimate for F, we easily obtain

$$\mathrm{L}_1 \leq C K^2_{Q,R}\|b_1\|_*\|b_2\|_* M^{(\alpha)}_{r,(9/8)} f(x). \tag{2.13}$$

We estimate F_2 by decomposing it into

$$\begin{aligned}\mathrm{L}_2 &\le \big|m_R\left(I_\alpha\left[(b_1-b_R^1)(b_2-b_R^2)f\chi_{\mathbb{R}^d\setminus 2^NQ}\right]\right)\\ &\qquad -m_R\left(I_\alpha\left[(b_1-b_R^1+b_R^1-b_Q^1)(b_2-b_R^2+b_R^2-b_Q^2)f\chi_{\mathbb{R}^d\setminus 2^NQ}\right]\right)\big|\\ &\le CK_{Q,R}\|b_2\|_*\left|m_R\left(I_\alpha\left[(b_1-b_R^1)f\chi_{\mathbb{R}^d\setminus 2^NQ}\right]\right)\right|\\ &\quad + CK_{Q,R}\|b_1\|_*\left|m_R\left(I_\alpha\left[(b_2-b_R^2)f\chi_{\mathbb{R}^d\setminus 2^NQ}\right]\right)\right|\\ &\quad + CK_{Q,R}^2\|b_1\|_*\|b_2\|_*\left|m_R\left[I_\alpha(f\chi_{\mathbb{R}^d\setminus 2^NQ})\right]\right|\\ &= \mathrm{M}_1+\mathrm{M}_2+\mathrm{M}_3.\end{aligned}$$

From the fact that R is a doubling cube, it is easy to deduce that

$$\mathrm{M}_3 \le CK_{Q,R}^2\|b_1\|_*\|b_2\|_* M_{r,(3/2)}\left[I_\alpha\left(|f|\right)\right](x). \tag{2.14}$$

We further estimate M_1 by writing

$$\begin{aligned}I_\alpha\left[(b_1-b_R^1)f\chi_{\mathbb{R}^d\setminus 2^NQ}\right](y) &= I_\alpha\left[(b_1-b_R^1)f\right](y) - I_\alpha\left[(b_1-b_R^1)f\chi_{2^NQ}\right](y)\\ &= (b_1(y)-b_R^1)I_\alpha(f)(y) - I_{\alpha;\,b_1}(f)(y)\\ &\quad - I_\alpha\left[(b_1-b_R^1)f\chi_{\frac{4}{3}R}\right](y)\\ &\quad - I_\alpha\left[(b_1-b_R^1)f\chi_{2^NQ\setminus\frac{4}{3}R}\right](y).\end{aligned}$$

From the fact that R is a doubling cube, it follows that

$$\frac{1}{\mu(R)}\int_R\left|b_1(y)-b_R^1\right|\left|I_\alpha(f)(y)\right|\,d\mu(y) \le C\|b_1\|_* M_{r,(3/2)}I_\alpha(f)(x)$$

and

$$\frac{1}{\mu(R)}\int_R\left|I_{\alpha;\,b_1}(f)(y)\right|\,d\mu(y) \le CM_{r,(3/2)}(I_{\alpha;\,b_1}f)(x).$$

An estimate similar to that for E and the fact that R is a doubling cube tell us that

$$\frac{1}{\mu(R)}\int_R\left|I_\alpha\left[(b_1-b_R^1)f\chi_{\frac{4}{3}R}\right](y)\right|\,d\mu(y) \le C\|b_1\|_* M^{(\alpha)}_{r,(9/8)}f(x).$$

Noting that $l(2^N Q) \sim l(R)$ and $2^N Q \supset R$, by the Hölder inequality and (2.1), for $y \in R$, we have

$$
\begin{aligned}
\left| I_\alpha \left[(b_1 - b_R^1) f \chi_{2^N Q \setminus \frac{4}{3} R} \right](y) \right| &= \left| \int_{2^N Q \setminus \frac{4}{3} R} \frac{b_1(z) - b_R^1}{|y-z|^{n-\alpha}} f(z)\, d\mu(z) \right| \\
&\le \frac{C}{l(R)^{n-\alpha}} \int_{2^N Q} \left| b_1(z) - b_R^1 \right| |f(z)|\, d\mu(z) \\
&\le \frac{C}{\mu(2^{N+1}Q)^{1-\alpha/n}} \left\{ \left[\int_{2^N Q} \left| b_1(z) - b_{2^N Q}^1 \right|^{r'} d\mu(z) \right]^{1/r'} \right. \\
&\qquad \left. + \left| b_{2^N Q}^1 - b_R^1 \right| \mu(2^N Q)^{1/r'} \right\} \\
&\qquad \times \left[\int_{2^N Q} |f(z)|^r \, d\mu(z) \right]^{1/r} \\
&\le C \|b_1\|_* \left[\frac{1}{\mu(2^{N+1}Q)^{1-\alpha r/n}} \int_{2^N Q} |f(z)|^r \, d\mu(z) \right]^{1/r} \\
&\le C \|b_1\|_* M_{r,(9/8)}^{(\alpha)} f(x).
\end{aligned}
$$

Thus

$$
\frac{1}{\mu(R)} \int_R \left| I_\alpha \left[(b_1 - b_R^1) f \chi_{2^N Q \setminus \frac{4}{3} R} \right](y) \right| d\mu(y) \le C \|b_1\|_* M_{r,(9/8)}^{(\alpha)} f(x).
$$

All the estimates above lead to

$$
\begin{aligned}
\mathrm{M}_1 &\le C K_{Q,R} \|b_1\|_* \|b_2\|_* \left\{ M_{r,(3/2)}(I_\alpha f)(x) + M_{r,(9/8)}^{(\alpha)}(f)(x) \right\} \\
&\quad + C K_{Q,R} \|b_2\|_* M_{r,(3/2)}(I_{\alpha;\, b_1} f)(x).
\end{aligned} \tag{2.15}
$$

Similarly, we can prove

$$
\begin{aligned}
\mathrm{M}_2 &\le C K_{Q,R} \|b_1\|_* \|b_2\|_* \left\{ M_{r,(3/2)}(I_\alpha f)(x) + M_{r,(9/8)}^{(\alpha)}(f)(x) \right\} \\
&\quad + C K_{Q,R} \|b_1\|_* M_{r,(3/2)}(I_{\alpha;\, b_2} f)(x).
\end{aligned} \tag{2.16}
$$

Combining the estimates (2.14), (2.15) and (2.16) leads to

$$\begin{aligned}\mathrm{L}_2 &\le C\|b_1\|_*\|b_2\|_* K_{Q,R}^2 \left\{ M_{r,(9/8)}^{(\alpha)} f(x) + M_{r,(3/2)}\left[I_\alpha(|f|)\right](x)\right\}\\ &\quad + C\|b_1\|_* K_{Q,R} M_{r,(3/2)}(I_{\alpha;\,b_2} f)(x)\\ &\quad + C\|b_2\|_* K_{Q,R} M_{r,(3/2)}(I_{\alpha;\,b_1} f)(x).\end{aligned} \tag{2.17}$$

Let us now estimate L_3 by first decomposing

$$\begin{aligned}&\left| I_\alpha \left[(b_1 - b_Q^1)(b_2 - b_Q^2) f \chi_{2^N Q \setminus \frac{4}{3}Q}\right](y)\right|\\ &\quad \le \left| I_\alpha \left[(b_1 - b_Q^1)(b_2 - b_Q^2) f \chi_{2Q \setminus \frac{4}{3}Q}\right](y)\right|\\ &\qquad + \left| I_\alpha \left[(b_1 - b_Q^1)(b_2 - b_Q^2) f \chi_{2^N Q \setminus 2Q}\right](y)\right|\\ &\quad \le C\|b_1\|_*\|b_2\|_* M_{r,(9/8)}^{(\alpha)} f(x)\\ &\qquad + C\sum_{k=1}^{N-1} \frac{1}{l(2^k Q)^{n-\alpha}} \int_{2^{k+1}Q\setminus 2^k Q} \left|(b_1(z) - b_Q^1)(b_2(z) - b_Q^2)\right| |f(z)|\, d\mu(z)\\ &\quad \le C\|b_1\|_*\|b_2\|_* M_{r,(9/8)}^{(\alpha)} f(x)\\ &\qquad + C K_{Q,R}^2 \|b_1\|_*\|b_2\|_*\\ &\qquad \times \sum_{k=1}^{N-1} \frac{\mu(2^{k+2}Q)^{1-\alpha/n}}{l(2^k Q)^{n-\alpha}} \left[\frac{1}{\mu(2^{k+2}Q)^{1-\alpha r/n}} \int_{2^{k+1}Q} |f(z)|^r\, d\mu(z)\right]^{1/r}\\ &\quad \le C K_{Q,R}^2 K_{Q,R}^{(\alpha)} \|b_1\|_*\|b_2\|_* M_{r,(9/8)}^{(\alpha)} f(x),\end{aligned}$$

where we used the Hölder inequality and (2.1). Taking the mean over Q, we then obtain

$$\mathrm{L}_3 \le C K_{Q,R}^2 K_{Q,R}^{(\alpha)} \|b_1\|_*\|b_2\|_* M_{r,(9/8)}^{(\alpha)} f(x). \tag{2.18}$$

Similarly, we have

$$\mathrm{L}_4 \le C\|b_1\|_*\|b_2\|_* M_{r,(9/8)}^{(\alpha)} f(x). \tag{2.19}$$

Combining the estimates (2.13), (2.17), (2.18) and (2.19) yields the estimate (2.5) and this finishes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We begin with some preliminary lemmas.

Lemma 3.1 ([**20**]). *Let $b \in \mathrm{RBMO}(\mu)$. There exist two positive constants C_1 and C_2 such that for any cube Q and $\lambda > 0$.*

$$\mu\left(\{x \in Q : |b(x) - m_{\widetilde{Q}}(b)| > \lambda\}\right) \le C_1 \mu(2Q) \exp\left(\frac{-C_2\lambda}{\|b\|_{\mathrm{RBMO}(\mu)}}\right).$$

The following lemma on the Calderón-Zygmund decomposition in non-homogeneous spaces can be found in [**22**], [**19**].

Lemma 3.2. *Assume that μ satisfies* (1.1)*. For any $f \in L^1(\mu)$ and any $\lambda > 0$ (with $\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$ if $\|\mu\| < \infty$), we have*

(a) *There exists a family of almost disjoint cubes $\{Q_j\}_j$ (that is, $\sum_j \chi_{Q_j}(x) \le C$) such that*

$$\frac{\lambda}{2^{d+1}} < \frac{1}{\mu(2Q_j)} \int_{Q_j} |f(x)|\, d\mu(x),$$

and

$$\frac{1}{\mu(2\eta Q_j)} \int_{\eta Q_j} |f(x)|\, d\mu(x) \le \frac{\lambda}{2^{d+1}}$$

for all $\eta > 2$.

(b) $|f(x)| \le \lambda$ *μ-a.e. on $\mathbb{R}^d \setminus \cup_j Q_j$.*

(c) *For each fixed j, let R_j be a $(6, 6^{n+1})$-doubling cube concentric with Q_j, with $l(R_j) > 4l(Q_j)$ and set $w_j = \chi_{Q_j}/\sum_k \chi_{Q_k}$. Then there exists a family of functions ϕ_j with $\operatorname{supp}\phi_j \subset R_j$ and with constant sign satisfying*

$$\int_{\mathbb{R}^d} \phi_j(x)\, d\mu(x) = \int_{Q_j} f(x) w_j(x)\, d\mu(x),$$

$$\|\phi_j\|_{L^\infty(\mu)} \mu(R_j) \le C \int_{Q_j} |f(x)|\, d\mu(x)$$

and

$$\sum_j |\phi_j(x)| \le C\lambda.$$

Proof of Theorem 1.2: To prove the theorem, without loss of generality, we may assume that $\|f\|_{L^1(\mu)} = 1$ and $\|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}(\mu) = 1$ for

$j=1,\ldots,m$. In fact, let

$$\widetilde{b}_j = \frac{b_j}{\|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}(\mu)}}$$

for $j=1,\ldots,m$. The homogeneity tells us that

$$\begin{aligned}(3.1)\quad &\mu\left(\left\{x\in\mathbb{R}^d : |I_{\alpha;\,b_1,\ldots,b_m}f(x)|>\lambda\right\}\right)\\ &= \mu\left(\left\{x\in\mathbb{R}^d : \left|I_{\alpha;\,\widetilde{b}_1,\ldots,\widetilde{b}_m}\left[\frac{f(x)}{\|f\|_{L^1(\mu)}}\right]\right|\right.\right.\\ &\qquad\left.\left. > \frac{\lambda}{\|f\|_{L^1(\mu)}\prod_{j=1}^m \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}(\mu)}}\right\}\right).\end{aligned}$$

Noting that

$$\left\|\frac{f}{\|f\|_{L^1(\mu)}}\right\|_{L^1(\mu)} = 1$$

and

$$\left\|\widetilde{b}_j\right\|_{\mathrm{Osc}_{\exp L^{r_j}}(\mu)} = 1$$

for $j=1,\ldots,m$, if the theorem is true when $\|f\|_{L^1(\mu)}=1$ and $\|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}(\mu)}=1$ for $j=1,\ldots,m$, by (3.1) and the inequalities

$$\Phi_s(t_1t_2)\le C\Phi_s(t_1)\Phi_s(t_2) \tag{3.2}$$

for any $s>0$, $t_1,t_2\ge 0$ and

$$\Phi_{1/r_\sigma}\left[\Phi_{1/r_{\sigma'}}(t)\right]\le C\Phi_{1/r}(t) \tag{3.3}$$

for $t\ge 0$, we easily deduce that the theorem still holds for any bounded f with compact support and any $b_j\in \mathrm{Osc}_{\exp L^{r_j}}(\mu)$ for $j=1,\ldots,m$.

In what follows, we prove the theorem by two steps.

Step I: In this step, we prove Theorem 1.2 for $m=1$.

For any fixed bounded and compact supported function f and any fixed

$$\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|,$$

applying the Calderón-Zygmund decomposition (see Lemma 3.2) to f at level λ^{q_0}, we obtain a sequence of cubes $\{Q_j\}_j$ with bounded overlaps, that is,

$$\sum_j \chi_{Q_j}(x)\le C<\infty,$$

such that

(I)

$$\frac{\lambda^{q_0}}{2^{d+1}} < \frac{1}{\mu(2Q_j)} \int_{Q_j} |f(x)|\, d\mu(x)$$

and

$$\frac{1}{\mu(2\eta Q_j)} \int_{\eta Q_j} |f(x)|\, d\mu(x) \le \frac{\lambda^{q_0}}{2^{d+1}}$$

for all $\eta > 2$.

(II) $|f(x)| \le \lambda^{q_0}$ μ-a.e. on $\mathbb{R}^d \setminus \cup_j Q_j$.

(III) For each fixed j, let R_j be the smallest $(6, 6^{n+1})$-doubling cube of the form $6^k Q_j$, $k \ge 1$. Set $w_j = \chi_{Q_j}/\sum_k \chi_{Q_k}$. Then there are a function ϕ_j with $\operatorname{supp}\phi_j \subset R_j$ and a constant $C > 0$ satisfying

$$\int_{\mathbb{R}^d} \phi_j(x)\, d\mu(x) = \int_{Q_j} f(x) w_j(x)\, d\mu(x),$$

$$\|\phi_j\|_{L^\infty(\mu)} \mu(R_j) \le C \int_{Q_j} |f(x)|\, d\mu(x)$$

and

$$\sum_j |\phi_j(x)| \le C\lambda^{q_0}.$$

It is easy to see that the conclusion of the theorem still holds if

$$\lambda \le 2^{d+1} \|f\|_{L^1(\mu)}/\|\mu\|$$

when $\|\mu\| < \infty$.

Decompose f into $f = g + h$, where

$$g(x) = f(x)\chi_{\mathbb{R}^d \setminus \cup_j Q_j}(x) + \sum_j \phi_j(x),$$

and

$$h(x) = f(x) - g(x) = \sum_j [w_j(x) f(x) - \phi_j(x)] = \sum_j h_j(x).$$

Let $1 < p_1 < n/\alpha$ and $1/q_1 = 1/p_1 - \alpha/n$. Recall that

$$\|g\|_{L^1(\mu)} \le C\|f\|_{L^1(\mu)} \le C$$

and $I_{\alpha;\, b_1}$ is bounded from $L^{p_1}(\mu)$ to $L^{q_1}(\mu)$ by Theorem 1.1 (see also [**2**]). This via the fact that $\|g\|_{L^\infty(\mu)} \le C\lambda^{q_0}$ gives

$$\begin{aligned}
\mu\left(\left\{x \in \mathbb{R}^d : |I_{\alpha;\, b_1} g(x)| > \lambda\right\}\right) &\le C\lambda^{-q_1} \int_{\mathbb{R}^d} |I_{\alpha;\, b_1} g(y)|^{q_1}\ d\mu(y) \\
&\le C\lambda^{-q_1} \|g\|_{L^{p_1}(\mu)}^{q_1} \\
&\le C\lambda^{-q_1} \lambda^{q_0(p_1-1)q_1/p_1} \|f\|_{L^1(\mu)}^{q_1/p_1} \\
&\le C\lambda^{-q_0}.
\end{aligned}$$

Noting that $r = r_1$ when $m = 1$ and

$$\mu\left(\bigcup_j 2Q_j\right) \le C\lambda^{-q_0} \int_{\mathbb{R}^d} |f(y)|\, d\mu(y) \le C\lambda^{-q_0},$$

therefore, the proof of the theorem can be reduced to proving that

$$\begin{aligned}
(3.4)\quad \mu&\left(\left\{x \in \mathbb{R}^d \setminus \bigcup_j 2Q_j : |I_{\alpha;\, b_1} h(x)| > \lambda\right\}\right) \\
&\le C\left[\left\|\Phi_{1/r}\left(\lambda^{-1}|f|\right)\right\|_{L^1(\mu)} + \Phi_{1/r}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right)\right]^{q_0}.
\end{aligned}$$

For each fixed j, set $b_1^{(j)}(x) = b_1(x) - m_{\widetilde{Q_j}}(b_1)$ and write

$$I_{\alpha;\, b_1} h(x) = \sum_j b_1^{(j)}(x) I_\alpha h_j(x) + I_\alpha(b_1^{(j)} h_j)(x) = \mathrm{I}(x) + \mathrm{I}I(x).$$

The weak type $(1, q_0)$ boundedness of I_α (see Proposition 6.2 in [**6**]) tells us that

$$\begin{aligned}
&\mu\left(\left\{x \in \mathbb{R}^d : |\mathrm{I}I(x)| > \lambda\right\}\right) \\
&\quad \le C\lambda^{-q_0}\left[\sum_j \int_{\mathbb{R}^d} |b_1^{(j)}(y)h_j(y)|\, d\mu(y)\right]^{q_0} \\
&\quad \le C\lambda^{-q_0}\left[\sum_j \int_{Q_j} |b_1(y) - m_{\widetilde{Q_j}}(b_1)||f(y)|\, d\mu(y)\right]^{q_0} \\
&\qquad + C\lambda^{-q_0}\left[\sum_j \|\phi_j\|_{L^\infty(\mu)} \int_{R_j} |b_1(y) - m_{\widetilde{Q_j}}(b_1)|\, d\mu(y)\right]^{q_0} \\
&\quad = \mathrm{U} + \mathrm{V}.
\end{aligned}$$

It is obvious that R_j is also $(2, \beta_d)$-doubling and $R_j = \widetilde{R_j}$. Thus, by Lemma 2.8 in [**20**],

$$\begin{aligned}
\int_{R_j} |b_1(y) - m_{\widetilde{Q_j}}(b_1)|\, d\mu(y) &\le \int_{R_j} |b_1(y) - m_{R_j}(b_1)|\, d\mu(y) \\
&\quad + \mu(R_j)\left[\left|m_{\widetilde{6Q_j}}(b_1) - m_{R_j}(b_1)\right|\right. \\
&\qquad \left. + \left|m_{\widetilde{6Q_j}}(b_1) - m_{\widetilde{Q_j}}(b_1)\right|\right] \\
&\le C\mu(2R_j) + C\mu(R_j)\left(K_{6Q_j,R_j} + K_{6Q_j,Q_j}\right).
\end{aligned}$$

A trivial computation shows that $K_{6Q_j,R_j} \le C$ (see also Lemma 2.1 (3) in [**20**]). This together with the estimate $\mu(2R_j) \le \mu(6R_j) \le 6^{n+1}\mu(R_j)$ in turn implies that

$$\mathrm{V} \le C\lambda^{-q_0}\left[\sum_j \|\phi_j\|_{L^\infty(\mu)}\mu(R_j)\right]^{q_0} \le C\lambda^{-q_0}\left[\int_{\mathbb{R}^d} |f(y)|\, d\mu(y)\right]^{q_0}, \tag{3.5}$$

which is a desired estimate for V.

On the other hand, by the generalized Hölder inequality (1.3) and Lemma 3.1, we have

$$\begin{aligned}
\mathrm{U} &\le C\lambda^{-q_0}\left\{\sum_j \mu(2Q_j)\|f\|_{L\log^{1/r}L, Q_j, \mu/\mu(2Q_j)}\|b_1^{(j)}\|_{\exp L^r, Q_j, \mu/\mu(2Q_j)}\right\}^{q_0} \\
&\le C\lambda^{-q_0}\left\{\sum_j \mu(2Q_j)\|f\|_{L\log^{1/r}L, Q_j, \mu/\mu(2Q_j)}\right\}^{q_0}.
\end{aligned}$$

From the fact that

$$\begin{aligned}
&\|f\|_{L\log^{1/r}L,Q_j,\mu/\mu(2Q_j)}\\
&\qquad \le \inf\left\{t+\frac{t}{\mu(2Q_j)}\int_{Q_j}\frac{|f(y)|}{t}\log^{1/r}\left(2+\frac{|f(y)|}{t}\right)\,d\mu(y)\right\}\\
&\qquad \le \lambda^{q_0}+\frac{1}{\mu(2Q_j)}\int_{Q_j}|f(y)|\log^{1/r}\left(2+\frac{|f(y)|}{\lambda^{q_0}}\right)\,d\mu(y)
\end{aligned}$$

(see [**16**] and the related references there) and the inequality

$$\log^{1/r}(2+t_1t_2)\le C\left[\log^{1/r}(2+t_1)+\log^{1/r}(2+t_1)\right]$$

for $t_1, t_2 \ge 0$, it follows that

(3.6)

$$\begin{aligned}
\mathrm{U}\le C\lambda^{-q_0}&\left\{\sum_j \mu(2Q_j)\lambda^{q_0}+\sum_j\int_{Q_j}|f(y)|\log^{1/r}\left(2+\frac{|f(y)|}{\lambda}\right)\,d\mu(y)\right.\\
&\qquad\qquad\left.+\sum_j\int_{Q_j}|f(y)|\,d\mu(y)\log^{1/r}\left(2+\frac{1}{\lambda}\right)\right\}^{q_0}\\
\le C&\left\{\frac{1}{\lambda}\|f\|_{L^1(\mu)}+\lambda^{-1}\log^{1/r}\left(2+\frac{1}{\lambda}\right)\|f\|_{L^1(\mu)}\right.\\
&\qquad\qquad\left.+\int_{\mathbb{R}^d}\frac{|f(y)|}{\lambda}\log^{1/r}\left(2+\frac{|f(y)|}{\lambda}\right)\,d\mu(y)\right\}^{q_0}\\
\le C&\left\{\Phi_{1/r}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right)+\left\|\Phi_{1/r}\left(\lambda^{-1}|f|\right)\right\|_{L^1(\mu)}\right\}^{q_0},
\end{aligned}$$

which is a desired estimate for U.

Combining (3.5) and (3.6), we obtain that

(3.7) $$\begin{aligned}
&\mu\left(\left\{x\in\mathbb{R}^d : |\mathrm{I}I(x)|>\lambda\right\}\right)\\
&\qquad\le C\left\{\Phi_{1/r}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right)+\left\|\Phi_{1/r}\left(\lambda^{-1}|f|\right)\right\|_{L^1(\mu)}\right\}^{q_0}.
\end{aligned}$$

Now we turn our attention to $\mathrm{I}(x)$. Denote by x_j the center of Q_j. Let θ be a bounded function with $\|\theta\|_{L^{q_0'}(\mu)} \le 1$ and the support contained in $\mathbb{R}^d \setminus \bigcup_j 2Q_j$. Write

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d\setminus\bigcup_j 2Q_j} \mathrm{I}(x)\theta(x)\,d\mu(x)\right|\\
&\le \sum_j \int_{\mathbb{R}^d\setminus 2R_j}\int_{\mathbb{R}^d} |b_1^{(j)}(x)\theta(x)| \left|\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x-x_j|^{n-\alpha}}\right| |h_j(y)|\,d\mu(y)\,d\mu(x)\\
&\quad + \sum_j \int_{2R_j\setminus 2Q_j} |b_1^{(j)}(x)\theta(x)|\,|I_\alpha h_j(x)|\;d\mu(x)\\
&\le C\sum_j l(Q_j)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus 2Q_j} \frac{|b_1^{(j)}(x)\theta(x)|}{|x-x_j|^{n+1-\alpha}} |h_j(y)|\,d\mu(y)\,d\mu(x)\\
&\quad + \sum_j \int_{2R_j\setminus 2Q_j} |b_1^{(j)}(x)\theta(x)|\,|I_\alpha(w_j f)(x)|\;d\mu(x)\\
&\quad + \sum_j \int_{2R_j} |b_1^{(j)}(x)\theta(x)||I_\alpha\phi_j(x)|\,d\mu(x)\\
&= \mathrm{G} + \mathrm{H} + \mathrm{J}.
\end{aligned}$$

Invoking the condition (1.1), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus 2Q_j} \frac{|b_1^{(j)}(x)\theta(x)|}{|x-x_j|^{n+1-\alpha}}\,d\mu(x)\\
&\le C\sum_{k=1}^{\infty}\left[2^k l(Q_j)\right]^{-n-1+\alpha}\int_{2^{k+1}Q_j}\left|b_1(x) - m_{\widetilde{2^{k+1}Q_j}}(b_1)\right| |\theta(x)|\,d\mu(x)\\
&\quad + C\sum_{k=1}^{\infty}\left[2^k l(Q_j)\right]^{-n-1+\alpha}\left|m_{\widetilde{Q_j}}(b_1) - m_{\widetilde{2^{k+1}Q_j}}(b_1)\right|\int_{2^kQ_j}\theta(x)\,d\mu(x)\\
&\le C\sum_{k=1}^{\infty}\left[2^k l(Q_j)\right]^{-n-1+\alpha}\left[\int_{2^{k+1}Q_j}\left|b_1(x) - m_{\widetilde{2^{k+1}Q_j}}(b_1)\right|^{q_0}\;d\mu(x)\right]^{1/q_0}\\
&\quad + C\sum_{k=1}^{\infty} K_{2^{k+1}Q_j,Q_j}\left[2^k l(Q_j)\right]^{-n-1+\alpha}\mu(2^{k+1}Q_j)^{1/q_0}\\
&\le C\sum_{k=1}^{\infty}\left[2^k l(Q_j)\right]^{-n-1+\alpha}\mu(2^{k+2}Q_j)^{1/q_0} + Cl(Q_j)^{-1}\\
&\le Cl(Q_j)^{-1}.
\end{aligned}$$

Therefore,

$$\mathrm{G} \le C \sum_j \|h_j\|_{L^1(\mu)} \le C \sum_j \int_{Q_j} |f(y)|\, d\mu(y) \le C \int_{\mathbb{R}^d} |f(y)| dy, \tag{3.8}$$

which is a desired estimate for G.

On the other hand, applying the Hölder inequality and the (L^{p_1}, L^{q_1}) boundedness of I_α (see Proposition 6.2 in [**6**]), we obtain

(3.9)

$$\begin{aligned}
\mathrm{J} &\le \sum_j \int_{2R_j} \left|b_1(x) - m_{\widetilde{2R_j}}(b_1)\right| |I_\alpha \phi_j(x)\theta(x)|\ d\mu(x) \\
&\quad + \sum_j \left|m_{\widetilde{Q_j}}(b_1) - m_{\widetilde{2R_j}}(b_1)\right| \int_{2R_j} |I_\alpha \phi_j(x)|\theta(x)||\ d\mu(x) \\
&\le \|\theta\|_{L^{q_0'}(\mu)} \left\{ \sum_j \left[\int_{2R_j} \left|b_1(x) - m_{\widetilde{2R_j}}(b_1)\right|^{q_0} |I_\alpha \phi_j(x)|^{q_0}\ d\mu(x) \right]^{1/q_0} \right. \\
&\quad \left. + \sum_j \left[\int_{2R_j} |I_\alpha \phi_j(x)|^{q_0}\ d\mu(x) \right]^{1/q_0} \left|m_{\widetilde{Q_j}}(b_1) - m_{\widetilde{2R_j}}(b_1)\right| \right\} \\
&\le \sum_j \left[\int_{2R_j} \left|b_1(x) - m_{\widetilde{2R_j}}(b_1)\right|^{q_0(q_1/q_o)'}\ d\mu(x) \right]^{1/q_0 - 1/q_1} \|I_\alpha \phi_j\|_{L^{q_1}(\mu)} \\
&\quad + \sum_j [\mu(4R_j)]^{1/q_0 - 1/q_1} \|I_\alpha \phi_j\|_{L^{q_1}(\mu)} \left|m_{\widetilde{Q_j}}(b_1) - m_{\widetilde{2R_j}}(b_1)\right| \\
&\le C \sum_j [\mu(4R_j)]^{1/q_0 - 1/q_1} \|I_\alpha \phi_j\|_{L^{q_1}(\mu)} \left[1 + \left|m_{\widetilde{Q_j}}(b) - m_{\widetilde{2R_j}}(b_1)\right|\right] \\
&\le C \sum_j [\mu(4R_j)]^{1/q_0 - 1/q_1} \|\phi_j\|_{L^{p_1}(\mu)} \\
&\le C \sum_j [\mu(4R_j)]^{1/q_0 - 1/q_1} \|\phi_j\|_{L^\infty(\mu)} [\mu(R_j)]^{1/p_1} \\
&\le C \int_{\mathbb{R}^d} |f(y)|\, d\mu(y),
\end{aligned}$$

where we have used the estimate that

$$\left|m_{\widetilde{Q_j}}(b_1)-m_{\widetilde{2R_j}}(b_1)\right| \le \left|m_{\widetilde{Q_j}}(b_1)-m_{R_j}(b_1)\right|+\left|m_{R_j}(b_1)-m_{\widetilde{2R_j}}(b_1)\right| \le C,$$

by recalling that R_j is $(2,\beta_d)$-doubling and $\widetilde{R_j}=R_j$.

To estimate H, observe that for $x\in 2R_j\backslash 2Q_j$,

$$|I_\alpha(w_jf)(x)|\le C\frac{1}{|x-x_j|^{n-\alpha}}\int_{Q_j}|f(y)|\,d\mu(y).$$

Write

$$\begin{aligned}
\mathrm{H}\le{}& C\sum_j\left\{\int_{2R_j\backslash R_j}\frac{|b_1^{(j)}(x)\theta(x)|}{|x-x_j|^{n-\alpha}}\,d\mu(x)\right.\\
&\left.+\int_{R_j\backslash Q_j}\frac{|b_1^{(j)}(x)\theta(x)|}{|x-x_j|^{n-\alpha}}\,d\mu(x)\right\}\int_{Q_j}|f(y)|\,d\mu(y)\\
\le{}& C\sum_j\frac{1}{l(R_j)^{n-\alpha}}\|\theta\|_{L^{q_0'}(\mu)}\left\{\int_{2R_j}|b_1^{(j)}(x)|^{q_0}\,d\mu(x)\right\}^{1/q_0}\int_{Q_j}|f(y)|\,d\mu(y)\\
&+C\sum_j\sum_{k=0}^{N-1}l(6^kQ_j)^{-n+\alpha}\int_{6^{k+1}Q_j\backslash 6^kQ_j}\left|b_1(x)-m_{\widetilde{6^{k+1}Q_j}}(b_1)\right||\theta(x)|\,d\mu(x)\\
&\times\int_{Q_j}|f(y)|\,d\mu(y)\\
&+C\sum_j\sum_{k=0}^{N-1}l(6^kQ_j)^{-n+\alpha}\left|m_{\widetilde{Q_j}}(b_1)-m_{\widetilde{6^{k+1}Q_j}}(b_1)\right|\\
&\times\int_{6^{k+1}Q_j}|\theta(x)|\,d\mu(x)\int_{Q_j}|f(y)|\,d\mu(y),
\end{aligned}$$

where N is the positive integer such that $R_j=6^NQ_j$. Obviously, for each $0\le k\le N-1$, $6^{k+1}Q_j\subset R_j$ and so

$$\left|m_{\widetilde{Q_j}}(b_1)-m_{\widetilde{6^{k+1}Q_j}}(b_1)\right|\le CK_{Q_j,6^{k+1}Q_j}\le CK_{Q_j,R_j}\le C,$$

which leads to that

$$
\begin{aligned}
\mathrm{H} \leq\, & C \sum_j \int_{Q_j} |f(y)|\, d\mu(y) \\
& + C \sum_j \sum_{k=0}^{N-1} l(6^k Q_j)^{-n+\alpha} \left[\mu(2 \times 6^{k+1} Q_j)\right]^{1/q_0} \int_{Q_j} |f(y)|\, d\mu(y) \\
& + C \sum_j \sum_{k=0}^{N-1} l(6^k Q_j)^{-n+\alpha} \left[\mu(6^{k+1} Q_j)\right]^{1/q_0} \int_{Q_j} |f(y)|\, d\mu(y) \\
\leq\, & C \sum_j \left\{ 1 + \sum_{k=0}^{N-1} l(6^k Q_j)^{-n+\alpha} \left[\mu(2 \times 6^{k+1} Q_j)\right]^{1/q_0} \right\} \int_{Q_j} |f(y)|\, d\mu(y).
\end{aligned}
$$

Note that there is no $(6, 6^{n+1})$-doubling cube between Q_j and R_j. It follows that for each integer k with $0 \leq k \leq N-1$,

$$
\begin{aligned}
\mu(6^{k+1} Q_j) &\leq \frac{\mu(6^N Q_j)}{6^{(n+1)(N-k-1)}} \\
&\leq C l(6^N Q_j)^n 6^{-(n+1)(N-k-1)} \\
&= C l(6^k Q_j)^n 6^{k-N}.
\end{aligned}
$$

We thus obtain that for each fixed j,

$$
\begin{aligned}
\sum_{k=0}^{N-1} l(6^k Q_j)^{-n+\alpha} \left[\mu(6^{k+2} Q_j)\right]^{1/q_0} &\leq C \sum_{k=1}^{N-1} l(6^{k-1} Q_j)^{-n+\alpha} \left[\mu(6^{k+1} Q_j)\right]^{1/q_0} \\
&\quad + l(6^{N-1} Q_j)^{-n+\alpha} \left[\mu(6 R_j)\right]^{1/q_0} \\
&\leq C \sum_{k=1}^{\infty} 6^{-k/q_0} + C \\
&\leq C.
\end{aligned}
$$

Thus,

$$
\mathrm{H} \leq C \sum_j \int_{Q_j} |f(y)|\, d\mu(y) \leq C \int_{\mathbb{R}^d} |f(y)|\, d\mu(y). \tag{3.10}
$$

Combining the estimates (3.8), (3.9) and (3.10) above yields

$$\left|\int_{\mathbb{R}^d\setminus\bigcup_j 2Q_j} \mathrm{I}(x)\theta(x)\,d\mu(x)\right| \le C\sum_j\int_{Q_j}|f(y)|\,d\mu(y) \le C\int_{\mathbb{R}^d}|f(y)|\,d\mu(y),$$

and so

$$\begin{aligned}\mu\left(\left\{x\in\mathbb{R}^d\setminus\bigcup_j 2Q_j : |\mathrm{I}(x)|>\lambda\right\}\right) &\le C\lambda^{-q_0}\int_{\mathbb{R}^d\setminus\bigcup_j 2Q_j}|\mathrm{I}(x)|^{q_0}\,d\mu(x)\\ &\le C\left[\lambda^{-1}\int_{\mathbb{R}^d}|f(y)|\,d\mu(y)\right]^{q_0}.\end{aligned}\tag{3.11}$$

The estimates (3.7) and (3.11) then yield (3.4), and we have completed the proof of the theorem for $m=1$.

Step II: In this step, we prove Theorem 1.2 for all $m\in\mathbb{N}$ by induction on m. To this end, we assume that $m\ge 2$ is an integer and that for any $1\le i\le m-1$ and any subset $\sigma=\{\sigma(1),\dots,\sigma(i)\}$ of $\{1,\dots,m\}$, Theorem 1.2 is true. We now verify that Theorem 1.2 is also true for m.

For each fixed f and $\lambda>0$, we decompose f by the same way as in Step I; see (I), (II) and (III) in the proof of Step I. And let Q_j, R_j, ϕ_j, w_j, g, h be the same as in Step I. To prove the theorem in this case, it suffices to verity

$$\begin{aligned}\mu\left(\left\{x\in\mathbb{R}^d\setminus\bigcup_j 2Q_j : |I_{\alpha;\,b_1,\dots,b_m}h(x)|>\lambda\right\}\right)\\ \le C\left\{\sum_{i=0}^{\infty}\sum_{\sigma\in C_i^m}\Phi_{1/r_{\sigma'}}\left[\left\|\Phi_{1/r_\sigma}(\lambda^{-1}|f|)\right\|_{L^1(\mu)}\right]\right\}^{q_0},\end{aligned}\tag{3.12}$$

where for $\sigma\in C_i^m$, $1/r_\sigma=1/r_{\sigma(1)}+\dots+1/r_{\sigma(1)}$ and $1/r_{\sigma'}=1/r-1/r_\sigma$.

For simplicity, we first introduce some notation. Let $\vec{b}=\{b_1,\dots,b_m\}$. For all $1\le i\le m$ and $\sigma=\{\sigma(1),\dots,\sigma(i)\}\in C_i^m$, we define $\vec{b}_\sigma=\{b_{\sigma(1)},\dots,b_{\sigma(i)}\}$ and set

$$I_{\alpha;\,b_{\sigma(1)},\dots,b_{\sigma(i)}}f(x)=I_{\alpha;\,\vec{b}_\sigma}f(x).$$

In particular, when $\sigma=\{1,\dots,m\}$, we denote $I_{\alpha;\,b_1,\dots,b_m}$ simply by $I_{\alpha;\,\vec{b}}$. For $1\le i\le m$ and all $\sigma\in C_i^m$, we write

$$[b(y)-b(z)]_\sigma=[b_{\sigma(1)}(y)-b_{\sigma(1)}(z)]\times\dots\times[b_{\sigma(i)}(y)-b_{\sigma(i)}(z)]$$

and

$$[m_{\widetilde{Q}}(b)-b(y)]_\sigma=[m_{\widetilde{Q}}(b_{\sigma(1)})-b_{\sigma(1)}(y)]\times\cdots\times[m_{\widetilde{Q}}(b_{\sigma(i)})-b_{\sigma(i)}(y)],$$

where Q is any cube in $\mathbb{R}^d$ and $y,z\in\mathbb{R}^d$. With the aid of the formula

$$\prod_{i=1}^m[m_{\widetilde{Q}}(b_i)-b_i(z)]=\sum_{i=0}^m\sum_{\sigma\in C_i^m}[b(y)-b(z)]_{\sigma'}[m_{\widetilde{Q}}(b)-b(y)]_\sigma$$

for $y,z\in\mathbb{R}^d$, where if $i=0$, then $\sigma'=\{1,\ldots,m\}$ and $\sigma=\emptyset$, it is easy to see

$$\begin{aligned}I_{\alpha;\vec{b}}h(x)&=\sum_j\sum_{i=1}^m\left[m_{\widetilde{Q}_j}(b_i)-b_i(x)\right]I_\alpha h_j(x)\\&\quad-\sum_j I_\alpha\left(\prod_{i=1}^m\left[m_{\widetilde{Q}_j}(b_i)-b_i\right]h_j\right)(x)\\&\quad-\sum_j\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}I_{\alpha;\vec{b}_{\sigma'}}\left(\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma h_j\right)(x)\\&=I^{\mathrm{I}}_{\alpha;\vec{b}}h(x)-I^{II}_{\alpha;\vec{b}}h(x)-\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}I^{III}_{\alpha;\vec{b}_{\sigma'}}h(x).\end{aligned}$$

An argument similar to that for $\mathrm{I}(x)$ in Step I gives us that

$$\mu\left(\left\{x\in\mathbb{R}^d\setminus\bigcup_j 2Q_j:|I^{\mathrm{I}}_{\alpha;\vec{b}}h(x)|>\lambda\right\}\right)\le C\left\{\lambda^{-1}\int_{\mathbb{R}^d}|f(y)|\,d\mu(y)\right\}^{q_0},\tag{3.13}$$

and an argument similar to that for $\mathrm{I}I(x)$ as in Step I yields

$$\begin{aligned}&\mu\left(\left\{x\in\mathbb{R}^d\setminus\bigcup_j 2Q_j:|I^{\mathrm{II}}_{\alpha;\vec{b}}h(x)|>\lambda\right\}\right)\\&\qquad\le C\left\{\Phi_{1/r}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right)+\left\|\Phi_{1/r}\left(\lambda^{-1}|f|\right)\right\|_{L^1(\mu)}\right\}^{q_0}.\end{aligned}\tag{3.14}$$

Now we estimate $I^{\mathrm{III}}_{\alpha;\vec{b}_{\sigma'}}h(x)$. For each fixed i with $1 \le i \le m-1$, the induction hypothesis now states that

$$\begin{aligned}
&\mu\left(\{x \in \mathbb{R}^d : |I^{\mathrm{III}}_{\alpha;\vec{b}_{\sigma'}}h(x)| > \lambda\}\right)\\
&\le C\left\{\sum_{l=0}^{m-i}\sum_{\eta\in C_l^{m-i}}\Phi_{1/r_{\eta'}}\left[\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}\sum_j\left|\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma h_j\right|\right)\right\|_{L^1(\mu)}\right]\right\}^{q_0}\\
&\le C\left\{\sum_{l=1}^{m-i}\sum_{\eta\in C_l^{m-i}}\Phi_{1/r_{\eta'}}\left[\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}\sum_j\left|\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma f\omega_j\right|\right)\right\|_{L^1(\mu)}\right]\right\}^{q_0}\\
&\quad + C\left\{\sum_{l=1}^{m-i}\sum_{\eta\in C_l^{m-i}}\Phi_{1/r_{\eta'}}\left[\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}\sum_j\left|\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma \phi_j\right|\right)\right\|_{L^1(\mu)}\right]\right\}^{q_0}\\
&\quad + C\left\{\Phi_{1/r_{\sigma'}}\left[\left\|\lambda^{-1}\sum_j\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma h_j\right\|_{L^1(\mu)}\right]\right\}^{q_0}\\
&= \mathrm{A}+\mathrm{B}+\mathrm{D}.
\end{aligned}$$

For A, we consider the following two cases.

Case I. $\lambda \ge 1$. Set $\Psi_r(t) = \exp t^r - 1$. Note

$$\Psi_r^{-1}(t) \sim \log^{1/r}(2+t),$$
$$\Phi_r^{-1}(t) \sim t\log^{-1/r}(2+t).$$

By Lemma 2.2 in [**16**], we see that for any $1 \le i \le m-1$, $\sigma \in C_i^m$ and any $t_0, t_1, \dots, t_i > 0$,

$$\Phi_{1/r_\eta}(t_0t_1\cdots t_i) \le C\left[\Phi_{1/r_\eta+1/r_\sigma}(t_0) + \Psi_{r_{\sigma(1)}}(t_1) + \cdots + \Psi_{r_{\sigma(i)}}(t_i)\right].$$

From this, it follows that

$$\begin{aligned}
&\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}\sum_j\left|\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma f\omega_j\right|\right)\right\|_{L^1(\mu)}\\
&\le C\sum_j\int_{Q_j}\Phi_{1/r_\eta}\left(\lambda^{-1}\left|f(x)\left[m_{\widetilde{Q}_j}(b)-b(x)\right]_\sigma\right|\right)\,d\mu(x)\\
&\le C\sum_j\int_{Q_j}\Phi_{1/r_\eta+1/r_\sigma}\left(\lambda^{-1}|f(x)|\right)\,d\mu(x)\\
&\quad+C\sum_j\sum_{l=1}^{i}\int_{Q_j}\Psi_{r_{\sigma(l)}}\left(\left|m_{\widetilde{Q}_j}(b_{\sigma(l)})-b_{\sigma(l)}(x)\right|\right)\,d\mu(x)\\
&\le C\sum_j\int_{Q_j}\Phi_{1/r_\eta+1/r_\sigma}\left(\lambda^{-1}|f(x)|\right)\,d\mu(x)+C\sum_j\mu(2Q_j)\\
&\le C\sum_j\int_{Q_j}\Phi_{1/r_\eta+1/r_\sigma}\left(\lambda^{-1}|f(x)|\right)\,d\mu(x)+C\lambda^{-q_0}\sum_j\int_{Q_j}|f(x)|\,d\mu(x)\\
&\le C\sum_j\int_{Q_j}\Phi_{1/r_\eta+1/r_\sigma}(\lambda^{-1}|f(x)|)\,d\mu(x)+C\lambda^{-1}\sum_j\int_{Q_j}|f(x)|\,d\mu(x)\\
&\le C\left\|\Phi_{1/r_\eta+1/r_\sigma}\left(\lambda^{-1}|f(x)|\right)\right\|_{L^1(\mu)}.
\end{aligned}$$

Case II. $0<\lambda<1$. In this case, we have

$$\begin{aligned}
&\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}\sum_j\left|\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma f\omega_j\right|\right)\right\|_{L^1(\mu)}\\
&\le C\sum_j\int_{Q_j}\Phi_{1/r_\eta}\left(\lambda^{q_0-1}\right)\Phi_{1/r_\eta}\left(\lambda^{-q_0}\left|f(x)\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma\right|\right)\,d\mu(x)\\
&\le C\sum_j\lambda^{q_0-1}\int_{Q_j}\Phi_{1/r_\eta+1/r_\sigma}(\lambda^{-q_0}|f(x)|)\,d\mu(x)+C\sum_j\lambda^{q_0-1}\mu(2Q_j)\\
&\le C\sum_j\int_{Q_j}\lambda^{-1}|f(x)|\log^{1/r_\eta+1/r_\sigma}\left(2+\lambda^{-q_0}|f(x)|\right)\,d\mu(x)\\
&\quad+C\lambda^{-1}\sum_j\int_{Q_j}|f(x)|\,d\mu(x)\\
&\le C\left\|\Phi_{1/r_\eta+1/r_\sigma}\left(\lambda^{-1}|f(x)|\right)\right\|_{L^1(\mu)}+\Phi_{1/r_\eta+1/r_\sigma}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right).
\end{aligned}$$

Thus we always have

$$\mathrm{A} \le C\left\{\sum_{l=1}^{m-i}\sum_{\eta\in C_l^{m-i}}\Phi_{1/r_{\eta'}}\left(\left\|\Phi_{1/r_\eta+1/r_\sigma}(\lambda^{-1}|f(x)|)\right\|_{L^1(\mu)}\right) + \Phi_{1/r}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right)\right\}^{q_0}. \tag{3.15}$$

Set $s_j(x)=\lambda^{-q_0}|\phi_j(x)|$ and denote by Λ a finite subset of $\mathbb{N}$. From (I), (II) and (III) of the decomposition of f, (3.2), the convexity of the function $\Phi_{1/r_\eta}(t)$ and the fact that $K_{R_j,\widetilde{Q_j}}\le C$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Phi_{1/r_\eta}\left\{\sum_{j\in\Lambda}\frac{|\phi_j(x)|}{\lambda}\left|\left[m_{\widetilde{Q_j}}(b)-b(x)\right]_\sigma\right|\chi_{R_j}(x)\right\}d\mu(x)\\
&\le C\Phi_{1/r_\eta}(\lambda^{q_0-1})\int_{\mathbb{R}^d}\Phi_{1/r_\eta}\left(\sum_{l\in\Lambda}s_l(x)\right)\Phi_{1/r_\eta}\\
&\qquad\times\left[\sum_{j\in\Lambda}\left(\frac{s_j(x)}{\sum_{l\in\Lambda}s_l(x)}\right)\left|\left[m_{\widetilde{Q_j}}(b)-b(x)\right]_\sigma\right|\chi_{R_j}(x)\right]d\mu(x)\\
&\le C\Phi_{1/r_\eta}\left(\lambda^{q_0-1}\right)\sum_{j\in\Lambda}\int_{\mathbb{R}^d}\Phi_{1/r_\eta}\left(\sum_{l\in\Lambda}s_l(x)\right)\left(\sum_{l\in\Lambda}s_l(x)\right)^{-1}s_j(x)\\
&\qquad\times\Phi_{1/r_\eta}\left\{\left|\left[m_{\widetilde{Q_j}}(b)-b(x)\right]_\sigma\right|\chi_{R_j}(x)\right\}d\mu(x)\\
&\le C\Phi_{1/r_\eta}(\lambda^{q_0-1})\sum_{j\in\Lambda}\int_{R_j}s_j(x)\left|\left[m_{\widetilde{Q_j}}(b)-b(x)\right]_\sigma\right|^2d\mu(x)\\
&\le C\Phi_{1/r_\eta}(\lambda^{q_0-1})\sum_{j\in\Lambda}\frac{\|\phi_j\|_{L^\infty(\mu)}}{\lambda^{q_0}}\mu(R_j)\left[1+\left(K_{R_j,\widetilde{Q_j}}\right)^2\right]\\
&\le C\Phi_{1/r_\eta}(\lambda^{q_0-1})\sum_{j\in\Lambda}\frac{\|\phi_j\|_{L^\infty(\mu)}}{\lambda^{q_0}}\mu(R_j)\\
&\le C\sum_{j\in\Lambda}\frac{1}{\lambda}\log^{1/r_\eta}(2+\lambda^{q_0-1})\|\phi_j\|_{L^\infty(\mu)}\mu(R_j)\\
&\le C\sum_{j\in\Lambda}\mu(Q_j)\frac{1}{\mu(Q_j)}\int_{Q_j}\frac{|f(x)|}{\lambda}d\mu(x)\log^{1/r_\eta}\left(2+\frac{1}{\mu(Q_j)}\int_{Q_j}\frac{|f(x)|}{\lambda}d\mu(x)\right)\\
&= C\sum_{j\in\Lambda}\mu(Q_j)\Phi_{1/r_\eta}\left[\frac{1}{\mu(Q_j)}\int_{Q_j}\frac{|f(x)|}{\lambda}d\mu(x)\right]\\
&\le C\int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log^{1/r_\eta}\left(2+\frac{|f(x)|}{\lambda}\right)d\mu(x) = C\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}|f|\right)\right\|_{L^1(\mu)},
\end{aligned}$$

where in the second-to-last step, we used the Jenson inequality, and the constant C is independent of Λ. From the arbitrariness of Λ, it follows that

$$\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}\sum_j\left|\left[m_{\widetilde{Q}_j}(b)-b\right]_\sigma\phi_j\right|\right)\right\|_{L^1(\mu)}\leq C\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}|f|\right)\right\|_{L^1(\mu)}.$$

Thus,

$$\mathrm{B}\leq C\left\{\sum_{l=1}^{m-i}\sum_{\eta\in C_l^{m-i}}\Phi_{1/r_{\eta'}}\left(\left\|\Phi_{1/r_\eta}\left(\lambda^{-1}|f|\right)\right\|_{L^1(\mu)}\right)\right\}^{q_0}. \tag{3.16}$$

An argument similar to that for $\mathrm{I}I(x)$ as in Step I yields

$$\begin{aligned}\mathrm{D}&\leq C\Big\{\Phi_{1/r_{\sigma'}}\left[\Phi_{1/r_\sigma}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right)\right]\\&\qquad+\Phi_{1/r_{\sigma'}}\left(\left\|\Phi_{1/r_\sigma}(\lambda^{-1}|f|)\right\|_{L^1(\mu)}\right)\Big\}^{q_0}\\&\leq C\left\{\Phi_{1/r}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right)+\Phi_{1/r_{\sigma'}}\left(\left\|\Phi_{1/r_\sigma}(\lambda^{-1}|f|)\right\|_{L^1(\mu)}\right)\right\}^{q_0}.\end{aligned} \tag{3.17}$$

Combining with (3.15), (3.16) and (3.17) tells us that

$$\begin{aligned}&\mu\left(\left\{x\in\mathbb{R}^d:|I^{III}_{\alpha;\vec{b}_{\sigma'}}h(x)|>\lambda\right\}\right)\\&\leq C\Bigg\{\sum_{l=0}^{m-i}\sum_{\eta\in C_l^{m-i}}\Phi_{1/r_{\eta'}}\left(\left\|\Phi_{1/r_\eta+1/r_\sigma}(\lambda^{-1}|f(x)|)\right\|_{L^1(\mu)}\right)\\&\qquad\qquad+\Phi_{1/r}\left(\lambda^{-1}\|f\|_{L^1(\mu)}\right)\Bigg\}^{q_0}.\end{aligned} \tag{3.18}$$

Finally, the estimates (3.13), (3.14) and (3.18) tell us the estimate (3.12), and we have completed the proof of Theorem 1.2. $\square$

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